Int. J. Solids Structures Vol. 28, No. 1, pp. 67–86, 1991
Printed in Great Britain.
0020-7683/91 $3.00 + .00

A MICROMECHANICAL-BASED MICROPOLAR THEORY FOR DEFORMATION OF GRANULAR SOLIDS

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(*Received 2 April* 1990; *in revised form* 27 *August* 1990)

Abstract—Granular material, perceived as a collection of particles, is modelled as a micropolar continuum taking account of the discreteness and microstructure of the material. The new feature of the proposed model is that the constitutive coefficients are derived in explicit terms of grain contact properties. In addition, the constitutive law, the equilibrium equations and the compatibility equations for granular material are derived to completely define a boundary value problem. Using the derived constitutive law and field equations, a procedure based on finite element analysis is described to obtain approximated solutions for boundary value problems. An example is shown for a granular packing under boundary pressure. Solutions of the example are compared with that obtained from the discrete element method to show the applicability of this method. Based on this model, the effects of particle rotation and couple stress on the deformation behavior of granular material are also studied.

1. INTRODUCTION

The mechanics of materials with rotations of the microbody is an old subject which can be dated back to Voigt (1887) and Cosserat and Cosserat (1968). More recently, considerable attention has been paid to the development of micropolar theories, for example, by Green (1965), Eringen (1968), Mindlin (1965), Nowacki (1986), Toupin (1964), etc. The theory of micropolar medium has been used to investigate the mechanical response of material in many areas such as the stress concentration around holes (Mindlin, 1962; Sternberg, 1968), the dynamics of composite materials (Herrmann and Achenbach, 1967) and wave propagation in half space (Ariman, 1972).

Although no experimental evidence has been found to reveal that the effect of microrotation is significant in metal, the theory does yield interesting phenomena and it has been suggested by several authors that the micropolar theory may be more applicable to granular material (Toupin, 1964; Eringen, 1968; Nowacki, 1986). In connection with the application of micropolar theory to the modelling of granular media, the micropolar theory has been found to be useful in the study of shear band thickness in bifurcation problem of granular material (Muhlhaus, 1989; Muhlhaus and Vardoulakis, 1987; Besdo, 1985).

However, in the above-mentioned studies, no consideration is given to account for the microscopic properties in the description of the constitutive behavior of the material. In view of the discrete nature of granular media, the primary mechanism of deformation and load carrying is the interactions at contacts between particles. Therefore it is desirable to have a theory that includes the micro measures. It is in need of, in the mathematical analysis, a transformation process through which the discrete system can be conceptually viewed as an equivalent continuum system.

This point of view has been paid very little attention in the study of material with microbody rotation. Work by Kanatani (1979) has been devoted to a continuum representation for granular flow where granular material is considered to be essentially liquid material. For the continuum representation of granular solids under a static equilibrium condition, efforts have been made to include the effect of particle interactions and grain contact properties (Chang and Liao, 1990).

In this paper, a model extended from the previous work (Chang, 1987; Chang and Liao, 1990) is proposed to represent the discrete granular material as equivalent continua.

In this model, we treat particle translation and particle rotation as two independent continuum fields based on the approach used in previous analyses. This leads to a generalized strain measure for the granular material that consists of both the deformation strain resulting from particle translations and the polar strain resulting from particle rotations. Corresponding to the strain measure, the stress measure consists of both Cauchy stress and polar stress.

Different from previous work (Chang and Liao, 1990), this current model defines stress and strain in reference to a particle point rather than an element comprising a large number of particles. The advantage of defining stress and strain at a particle point is that, for the small region containing this particle and its surrounding particles, the displacement and rotation fields can reasonably be assumed affine (i.e. homogeneous). This assumption avoids the use of complex higher-order kinematic fields such as that used in the previous model.

Another advantage is that, by defining stress and strain at a particle point, equilibrium equations can be expressed in a simple differential form instead of a complicated integral form corresponding to the higher-order kinematic field used in the previous study. The simpler form enables one to derive the constitutive law and field equations with explicit expressions of microscopic properties. Thus a boundary value problem for granular material is completely defined.

Using the derived constitutive law and field equations, a procedure based on finite element analysis is used to solve boundary value problems. An example is shown for a granular packing under boundary pressures. Results of the example are compared with that obtained from the discrete element method to show the applicability of this method. A discussion is given on the deformation behavior in relation to particle rotation and couple stress.

The model in this paper, for simplicity, is limited to the conditions of elastic interactions between particles. No separation, sliding and rearrangement between particles are allowed. The model, although under idealized elastic conditions, demonstrates that the constitutive parameters can be rationally derived based on microscopic properties. It also shows that particle rotation and couple stress play significant roles in the overall deformation behavior of granular material.

2. CONTINUUM MODELING OF DISCRETE GRANULAR MATERIAL

2.1. *Mechanics of interactions between particles*

For materials such as soil and ceramic at ordinary temperatures, granules are relatively rigid and the deformation of granules occurs mostly at contacts. Therefore we envisage a simple conceptual model in which the constituent particles are treated as rigid bodies. The particle motion consists of a translation and a rotation about the particle center. The pair of particles in contact are viewed to be connected at contact points by imaginary springs. In general, these springs are elasto-plastic: the elastic portion of the spring deformation represents a particle distortion while the plastic portion of the spring deformation represents a yielding at contacts between the particles. In this paper, for simplicity, only the elastic springs are considered.

To represent the contact resistance, two types of springs are used, namely the rotation springs and the stretch springs. The rotation springs, transmitting contact moments, represent the contact resistance to the relative rotation of two particles. The stretch springs, transmitting contact forces, represent the contact resistance to the relative translation of two particles.

The local kinematics of two particles in contact is schematically shown in Fig. 1. When the assembly deforms, particles undergo translations and rotations which result in the straining of springs at contacts. For example, as shown in Fig. 1, the translation and rotation are denoted as u_i and ω_i for a particle which is in contact with several particles. The movements of particle c in contact with this particle are denoted by u_i^c and ω_i^c. The angular straining, θ_i^c, of the rotation spring at contact point C is caused by the relative rotation of the two contact particles.

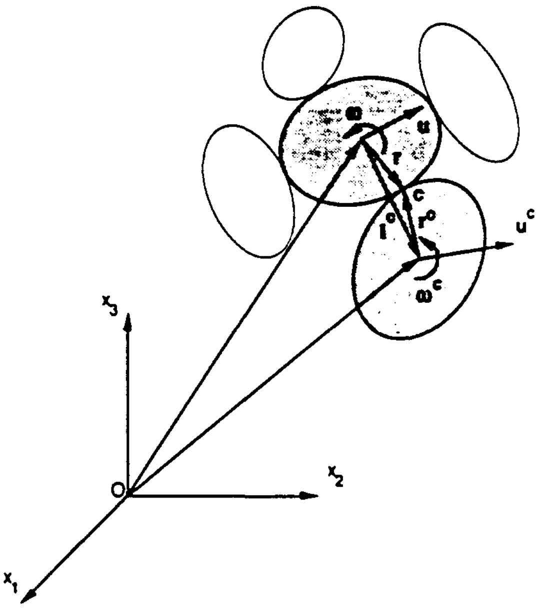


Fig. 1. Schematic illustration of particle interaction.

$$\theta_i^c = \omega_i^c - \omega_i . \tag{1}$$

The straining of the stretch springs, δ_i^c, is caused by the relative displacement at the contact point c of the two particles, expressed as:

$$\delta_i^c = (u_i^c - u_i) + \Xi_{ijk}(\omega_j^c r_k^c - \omega_j r_k) \tag{2}$$

where r_k and r_k^c, as shown in Fig. 1, are the vectors joining respectively the centroid of the reference particle and the centroid of the neighbor particle to the contact point. The quantity Ξ_{ijk} is the permutation symbol used in the tensor representation for the cross product of vectors.

The behavior of the springs at the contact point connecting two particles can be described by the relationships between the contact rotation, $\mathrm{d}\theta_j^c$, and the contact moment, $\mathrm{d}m_i^c$, i.e.

$$\mathrm{d}m_i^c = g_{ij}^c \,\mathrm{d}\theta_j^c \tag{3}$$

and between the contact deformation, $\mathrm{d}\delta_j^c$, and the contact force, $\mathrm{d}f_i^c$, i.e.

$$\mathrm{d}f_i^c = k_{ij}^c \,\mathrm{d}\delta_j^c \tag{4}$$

where g_{ij}^c is the stiffness tensor of the rotation spring, and k_{ij}^{cp} is the stiffness tensor of the stretch spring.

Let g_n^c, g_s^c and g_t^c be the rotational stiffness constants in the directions of the local coordinates n, s and t, respectively. The local coordinate system is constructed for each contact with three orthogonal base unit vectors; the vector **n** is normal to, and the vectors **s** and **t** are tangential to, the contact area. Thus

$$g_{ij}^{cr} = g_n^c n_i^c n_j^c + g_s^c s_i^c s_j^c + g_t^c t_i^c t_j^c . \tag{5}$$

Similarly, k_{ij}^c can be given by

$$k_{ij}^{cc} = k_n^c n_i^c n_j^c + k_s^c s_i^c s_j^c + k_t^c t_i^c t_j^c \tag{6}$$

where k_n^c, k_s^c and k_t^c are the spring constants in the directions of local coordinate system n, s and t, respectively.

Equations (3) and (4) can be combined into a compact form as follows:

$$d\{F_i^c\} = [K_{ij}^c]d\{D_j^c\} \tag{7}$$

where $\{D_i^c\}$ is the generalized contact-displacement vector, $\{F_i^c\}$ is the generalized contact-force vector and $[K_{ij}^c]$ is the generalized stiffness tensor, defined in the following:

$$\{D_i^c\} = \begin{Bmatrix} \delta_i^c \\ \theta_i^c \end{Bmatrix}; \quad \{F_i^c\} = \begin{Bmatrix} f_i^c \\ m_i^c \end{Bmatrix}; \quad [K_{ij}^c] = \begin{bmatrix} k_{ij}^{cc} & 0 \\ 0 & g_{ij}^{cc} \end{bmatrix}. \tag{8}$$

2.2. *Generalized strain tensor in a granular solid*

Let the centroid of a particle be denoted as a particle point. A granular material can thus be represented by a set of discrete particle points and the translations and rotations are discrete variables. However, when we deal with a small volume consisting of a large number of particles, these particle points can be conceptually viewed as continuous in the macroscopic scale. Thus the displacements and rotations can be represented by two independent continuum fields. In these continuum fields, at the neighborhood of a particle point, we assume the usual "affine" (or homogeneous) deformation. The displacement u_i^c and rotation ω_i^c at the adjacent particle point C (as in Fig. 1) can be expressed as:

$$u_i^c = u_i + u_{i,j} l_j^c \tag{9}$$

$$\omega_i^c = \omega_i + \omega_{i,j} l_j^c \tag{10}$$

where l^c is the branch vector joining the particle point and its adjacent particle point C, and $u_{i,j}$ and $\omega_{i,j}$ are the derivatives of the displacement and the rotation, respectively.

Substituting eqns (9) and (10) into eqns (1) and (2), the relative angular rotation, θ_i^c, and the deformation of the stretch-springs, δ_i^c, can be expressed by the derivative quantities $\omega_{i,j}$ and $u_{i,j}$, given by

$$\theta_i^c = \omega_{i,j} l_j^c \tag{11}$$

$$\delta_i^c = (u_{i,j} - \Xi_{jik}\omega_k) l_j^c + \Xi_{ijk}\omega_{j,l} l_l^c r_k^c. \tag{12}$$

Observing the expressions of eqns (11) and (12) we introduce, similar to the form used in theory of micropolar medium, the asymmetric deformation strain ε_{ji} and the polar strain γ_{ji} at the particle point

$$\varepsilon_{ji} = u_{i,j} - \Xi_{jik}\omega_k \tag{13}$$

$$\gamma_{ji} = \omega_{i,j}. \tag{14}$$

The symmetrical part of ε_{ji} is equal to the symmetrical part of the displacement gradient, i.e.

$$\varepsilon_{(ji)} = u_{(j,i)} = \tfrac{1}{2}(u_{j,i} + u_{i,j}) \tag{15}$$

representing the stretch strain. The skew-symmetric part of ε_{ji} is given by

$$\varepsilon_{[ji]} = u_{[j,i]} + \Xi_{jim}\omega_m \tag{16}$$

where the skew-symmetric part of the displacement gradient $u_{[j,i]} = \frac{1}{2}(u_{j,i} - u_{i,j})$ representing the rigid body rotation. The angular rotation ψ_m corresponding to the rigid body rotation is

$$\Xi_{jim}\psi_m = -u_{[j,i]}. \tag{17}$$

Thus the skew-symmetric part of ε_{ji} becomes

$$\varepsilon_{[ji]} = \Xi_{jim}(\omega_m - \psi_m) \tag{18}$$

which represents the net particle spin (i.e. the difference between the particle rotation and the rigid body rotation of the material).

Combining tensor and matrix notation, the above equations can be written in a compact form as follows:

$$\{E_{ji}\} = [\nabla_{jik}]\{U_k\} \tag{19}$$

where

$$\{E_{ji}\} = \begin{Bmatrix} \varepsilon_{ji} \\ \gamma_{ji} \end{Bmatrix}, \quad \{U_k\} = \begin{Bmatrix} u_k \\ \omega_k \end{Bmatrix}, \quad [\nabla_{jik}] = \begin{bmatrix} \partial_j\delta_{ik} & -\Xi_{jik} \\ 0 & \partial_j\delta_{ik} \end{bmatrix} \tag{20}$$

where $\partial_j = \partial/\partial x_j$ and δ_{ij} is the Kronecker delta, $\{E_{ji}\}$ represents the generalized strain tensor, $\{U_k\}$ is the generalized displacement vector and $[\nabla_{jik}]$ is the gradient operator. The summation convention of indices is followed in accordance with the tensor notation.

We further express the relative movement of a pair of contact particles [eqns (11), (12)] in terms of the generalized strain tensor $\{E_{ji}\}$ as follows:

$$\{D_i^c\} = [L_{ijk}^c]\{E_{jk}\} \tag{21}$$

where $\{L_i^c\}$, the same as that in eqn (8), is the generalized relative movement vector at the contact, and $[L_{ijk}^c]$ is the fabric operator, given by

$$[L_{ijk}^c] = \begin{bmatrix} \delta_{ik}l_j^c & \Xi_{ikl}l_j^c r_l^c \\ 0 & \delta_{ik}l_j^c \end{bmatrix}; \quad \{D_i^c\} = \begin{Bmatrix} \delta_i^c \\ \theta_i^c \end{Bmatrix}. \tag{22}$$

2.3. *Stresses in a granular assembly*

Corresponding to the strain tensor defined in eqn (19), the stress tensor in the granular assembly can be defined by using the principle of energy equivalence. in a granular assembly, the energy at a contact c of two particles due to an increment of load can be expressed as:

$$\mathrm{d}W^c = \{F_i^c\}^T \mathrm{d}\{D_i^c\}. \tag{23}$$

Assuming the energy at this contact is equally shared by the two contact particles, the total energy shared by particle "a" is the energy summed over all contacts with its surrounding particles, given by

$$\mathrm{d}W^a = \frac{1}{2}\sum_c \mathrm{d}W^c. \tag{24}$$

The energy thus defined is discrete quantities for each particle. In a continuum system, it is desirable to define the energy as a quantity, in the units of energy per volume, continuously distributed in the material. Let V^a be the volume associated with particle "a" which includes the volume of the solid particle v^a, and its associated voids. It is convenient to express the particle volume V^a in terms of the porosity n of the assembly by

$$V^a = v^a/n \tag{25}$$

where the porosity n is the ratio of the total volume of the assembly to the total volume of solid particles within the assembly, given by

$$n = \sum \frac{v^a}{V}. \tag{26}$$

This satisfies the requirement that the total volume of the assembly is the sum of all particle volumes, i.e.

$$V = \sum_a V^a.$$

We assume the energy for particle "a", $\mathrm{d}W^a$, is equally distributed in the volume associated with the particle, V^a. Thus the energy per volume $\mathrm{d}\bar{W}$ in the volume associated with particle "a" is given by

$$\mathrm{d}\bar{W} = \frac{\mathrm{d}W^a}{V^a} = \frac{1}{2V^a}\sum_c \mathrm{d}W^c. \tag{27}$$

For small strain condition, by substituting eqn (21) into (23), $\mathrm{d}W^c$ can be expressed in terms of strain $\mathrm{d}\{E_{jk}\}$ as follows:

$$\mathrm{d}\bar{W} = \left\{\frac{1}{2V^a}\sum_c \{F_i^c\}^{\mathrm{T}}[L_{ijk}^c]\right\}\mathrm{d}\{E_{jk}\}. \tag{28}$$

For a micropolar medium, the energy per volume can also be expressed in terms of stress and strain, i.e.

$$\mathrm{d}\bar{W} = \{S_{jk}\}^{\mathrm{T}}\{\mathrm{d}E_{jk}\} \tag{29}$$

where $\{S_{jk}\}$ is the generalized stress tensor, given by

$$\{S_{jk}\} = \begin{Bmatrix}\sigma_{jk}\\ \mu_{jk}\end{Bmatrix}, \tag{30}$$

in which σ_{jk} is the Cauchy stress tensor and μ_{jk} the polar stress tensor referred to a particle point.

Comparing eqns (28) and (29), one can define the generalized stress tensor of the granular medium in terms of generalized forces vector and fabric tensor operator; it follows that

$$\{S_{jk}\} = \frac{1}{2V^a}\sum_c [L_{ijk}^c]^{\mathrm{T}}\{F_i^c\}. \tag{31}$$

Equation (31) can be expanded into two equations corresponding to Cauchy stress and couple stress as

$$\sigma_{jk} = \frac{1}{2V^{a}}\sum_{c} l_j^c f_k^c \tag{32}$$

$$\mu_{jk} = \frac{1}{2V^{a}}\sum_{c} l_j^c (m_k^c + \Xi_{ikl} r_l^c f_i^c). \tag{33}$$

The derived Cauchy stress in eqn (32), in terms of contact forces, has a form similar to that proposed by Christoffersen *et al.* (1981). Equation (33) defines couple stress in terms of contact forces, contact couples and microstructural measures l_j^c and r_l^c. Equation (33) reveals that the couple stress may be transmitted across a grain contact solely by contact forces. Therefore the couple stress, contrary to common preconception, can exist in a packing of particles in the absence of force couples on the contacts, such as in a packing of rigid smooth spheres which have no rotational resistance at contacts.

2.4. *Constitutive law of granular medium*

According to eqn (31), the incremental form of the stress tensor can be written as follows:

$$\mathrm{d}\{S_{ij}\} = \frac{1}{2V^{a}}\sum_{c} [L_{mij}^c]^{\mathrm{T}}\, \mathrm{d}\{F_m^c\}. \tag{34}$$

Substituting eqns (7) and (21) into (34), the constitutive law in an incremental form can be established to describe the relationships between stress and strain, given by

$$\mathrm{d}\{S_{ij}\} = [A_{ijkl}]\,\mathrm{d}\{E_{kl}\} \tag{35}$$

where $[A_{ijkl}]$ is the constitutive coefficient tensor, given by

$$[A_{ijkl}] = \frac{1}{2V^{a}}\sum_{c} [L_{mij}^c]^{\mathrm{T}}[K_{mn}^c][L_{nkl}^c]. \tag{36}$$

Expressed in detailed form, eqn (35) becomes

$$\begin{Bmatrix} \mathrm{d}\sigma_{ij} \\ \mathrm{d}\mu_{ij} \end{Bmatrix} = \begin{bmatrix} a_{ijkl} & b_{ijkl} \\ b_{klij} & c_{ijkl} \end{bmatrix} \begin{Bmatrix} \mathrm{d}\varepsilon_{kl} \\ \mathrm{d}\gamma_{kl} \end{Bmatrix} \tag{37}$$

where a_{ijkl}, b_{ijkl} and c_{ijkl} are the constitutive coefficients of eqn (36), given by

$$a_{ijkl} = \frac{1}{2V^{a}}\sum_{c} l_i^c K_{jl}^c l_k^c \tag{38}$$

$$b_{ijkl} = \frac{1}{4V^{a}}\sum_{c} \Xi_{lmn} l_i^c K_{jn}^c l_k^c r_m^c \tag{39}$$

$$c_{ijkl} = \frac{1}{2V^{a}}\sum_{c} l_i^c l_k^c (G_{jl}^c + \Xi_{jnq}\Xi_{lmp} K_{nm}^c r_p^c r_q^c). \tag{40}$$

Observing from eqns (38) and (40), it is noted that a_{ijkl} and c_{ijkl} have the following properties:

$$a_{ijkl} = a_{klij}; \quad c_{ijkl} = c_{klij}. \tag{41}$$

The constitutive coefficients are functions of the microstructure at a particle point.

According to eqn (39), it can be seen that $b_{ijkl} = 0$ when the microstructure is center-symmetric. Under a center-symmetric condition, the constitutive equations are decoupled as follows:

$$d\sigma_{ij} = a_{ijkl}\, d\varepsilon_{kl}; \tag{42}$$

$$d\mu_{ij} = c_{ijkl}\, d\varepsilon_{kl}. \tag{43}$$

When the couple stress tensor μ_{ij} and the polar strain γ_{kl} are neglected, the constitutive equation reduces to the usual form for a non-polar medium.

Based on eqn (40), the coefficients c_{ijkl} are contributed to by two sources: $K_{nm}r_p r_q$ related to stretch springs and G_{jl} related to rotation springs. The former is a function of particle size. Therefore, for packings with small size particles, the contribution of the stretch springs to the coefficient c_{ijkl} is relatively insignificant compared to that of the rotational springs.

2.5. *Equilibrium equations*

In this section, the governing equations are derived for a granular assembly in static equilibrium subjected to an external traction ($\bar{T}_i$) on the boundary surface of the assembly. Based on the principle of virtual work, the work done by the external traction is equal to the work done by the internal stresses. Therefore, one can write

$$\int_V \delta\{E_{ij}\}^T\{S_{ij}\}\, dV = \int_S \delta\{U_i\}^T\{\bar{T}_i\}\, dS. \tag{44}$$

Replacing the term $\delta\{E_{ij}\}$ on the left-hand side of eqn (44) by the gradients $\delta([\nabla_{ijk}]\{U_K\})$ and using the Gauss theorem, it follows that

$$\int_S \delta\{U_j\}^T\{S_{ij}\}n_i\, dS - \int_V \delta\{U_j\}^T[\nabla_{ijk}]^T\{S_{ik}\}\, dV = \int_S \delta\{U_i\}^T\{\bar{T}_i\}\, dS. \tag{45}$$

Equation (45) is arranged into the following expression:

$$\int_V \delta\{U_j\}^T([\nabla_{ijk}]^T\{S_{ik}\})\, dV + \int_S \delta\{U_j\}^T(\{\bar{T}_j\} - n_i\{S_{ij}\})\, dS = 0. \tag{46}$$

Since the choice of $\delta\{U_j\}$ and the choice of V and S in eqn (46) are arbitrary, the following equation must hold:

$$[\nabla_{ijk}]^T\{S_{ik}\} = 0. \tag{47}$$

Equation (47) can be expanded into two equilibrium equations, given by

$$\partial_i\sigma_{ij} = 0. \tag{48}$$

$$\partial_i\mu_{ij} - \Xi_{ijk}\sigma_{ik} = 0. \tag{49}$$

For the same reason, in eqn (46), the choice of $\delta\{U_j\}$ and the choice of V and S are arbitrary. On the boundary surface, it must satisfy the following two boundary conditions:

$$\{\bar{T}_j\} - n_i\{S_{ij}\} = 0 \quad \text{or} \quad \{U_j\} = \{\bar{U}_j\} \quad (\text{i.e. } \delta\{U_j\} = 0). \tag{50}$$

The general displacement vector $\{U_i\}$ consists of six variables. However, the functions $\{E_{ij}\}$ in eqn (19) represent 18 differential equations. The functions $\{E_{ij}\}$ therefore cannot be arbitrary and must be subject to certain restrictions which are called conditions of geometric compatibility. The conditions of geometric compatibility for $\{E_{ij}\}$ are derived to be

$$\Xi_{mjl}[\nabla_{jik}]\{E_{lk}\} = \Xi_{mjl}[\nabla_{jik}][\nabla_{lkn}]\{U_n\} = 0. \tag{51}$$

The above equation can also be expressed as follows:

$$[\nabla_{jik}]\{E_{lk}\} - [\nabla_{lik}]\{E_{jk}\} = 0. \tag{52}$$

3. SOLUTION BY MICROSTRUCTURAL FINITE ELEMENT METHOD

We next describe a method of solution for boundary value problems of granular material based on the continuum model discussed in the previous section. The method of solution, although utilizing similar techniques, is different from the conventional finite element method. The distinct difference is in the derivation of the stiffness matrix by considering the microstructural properties of the material. The method of solution, hereinafter, is referred to as the microstructural finite element method (MFEM).

In this method, each element consists of several nodal points which may be located on either the boundary or within the domain. Let $\{U_i^p\}$ be the generalized displacement vector of the nodal point "p". The displacement/rotation at any point (x_1, x_2, x_3) in the element can be expressed by an interpolation function $\Phi^p(x_1, x_2, x_3)$ and the displacement $\{U_i^p\}$ of the nodal points; that is,

$$\{U_i(x_1, x_2, x_3)\} = \sum_{p=1}^{N} \Phi^p(x_1, x_2, x_3)\{U_i^p\} \tag{53}$$

where N is the total number of nodal points. The coefficient $\Phi^p(x_1, x_2, x_3)$ is regarded as the weighting of nodal point "p" to point (x_1, x_2, x_3) in the element and it is usually constructed as polynomials.

Applying the principle of minimum potential energy on an element of the granular medium, we have

$$\delta\Pi = 0 = \int_V \delta\{E_{ij}\}^{\mathrm{T}}\{S_{ij}\}\,\mathrm{d}V - \int_S \delta\{U_i\}^{\mathrm{T}}\{\tilde{T}_i\}\,\mathrm{d}S. \tag{54}$$

Substituting eqns (19) and (53) into eqn (54) to express $\delta\{E_{ij}\}$ and $\delta\{U_i\}$ in terms of nodal displacements $\delta\{U_i^p\}$, eqn (54) becomes

$$\delta\Pi = 0 = \sum_{p=1}^{N} \delta\{U_k^p\}^{\mathrm{T}}\left(\int_V [\nabla_{ijk}]^{\mathrm{T}}\Phi^p\{S_{ij}\}\,\mathrm{d}V - \int_S \Phi^p\{\tilde{T}_k\}\,\mathrm{d}S\right). \tag{55}$$

Since the variation $\delta\{U_k^p\}$ in eqn (55) is arbitrary, the following equation must hold:

$$\int_V [\nabla_{ijk}]^{\mathrm{T}}\Phi^p\{S_{ij}\}\,\mathrm{d}V - \int_S \Phi^p\{\tilde{T}_k\}\,\mathrm{d}S = 0. \tag{56}$$

Note that $p = 1, \ldots, N$. Equation (56) represents a set of N simultaneous equations, N being the total number of nodal points of an element. Furthermore, by expressing the stress tensor $\{S_{ij}\}$ in terms of nodal displacement and writing the equation in an abbreviated matrix form, eqn (56) becomes

$$\sum_{q=1}^{N} [C_{kn}^{pq}]\{U_n^q\} = \{D_k^p\} \tag{57}$$

where $p = 1, \ldots, N$, and eqn (57) represents a set of N simultaneous equations. The summation convention for subscripts in tensor notation is followed. $[C_{kn}^{pq}]$ is the element stiffness matrix, and $\{D_k^p\}$ is a vector including nodal force and nodal couples, given by

$$[C_{kn}^{pq}] = \int_V [\nabla_{ijk}]^T \Phi^p [A_{ijlm}][\nabla_{lmn}]\Phi^q \, dV \tag{58}$$

$$\{D_k^p\} = \int_S \Phi^p \{\tilde{T}_k\} \, dS. \tag{59}$$

The integral given in eqn (58) over the volume of the assembly V can be carried over each particle. Since each particle is small, the integral over the particle volume can be approximated by multiplying the integrand value at the centroid of the particle by the volume associated with the particle. After the element stiffness matrix is established, boundary value problems can be solved using the usual finite element procedure.

The solution thus obtained gives a displacement field in accordance with the values of nodal displacement and the assumed interpolation function. Therefore the solution does not guarantee equilibrium of each particle. However, the solution meets the requirement of minimum potential energy for each element.

4. EXAMPLES OF TWO-DIMENSIONAL GRANULAR MATERIAL PACKINGS

Deformation behavior of a disk packing under external load is illustrated here to show applicability of the proposed theory. The structure of the random packing of circular disks, shown in Fig. 2, is obtained by digitizing a photograph of an assembly of aluminum rods randomly placed in a box of 7 in. (17.8 cm) by 8.1 in. (20.6 cm). The radius of each rod is 0.25 in. (0.64 cm) with the total number of rods being 276 and the total number of contacts 695. The elastic spring constants at the contacts between disks are assumed to be as follows: $K_n^c = 1000$ lb in.$^{-1}$ (8.85 kN m^{-1}); $K_s^c = 400$ lb in.$^{-1}$ (3.54 kN m^{-1}); and $G_r^c = 100$ lb in.$^{-1}$ (0.89 kN m^{-1}).

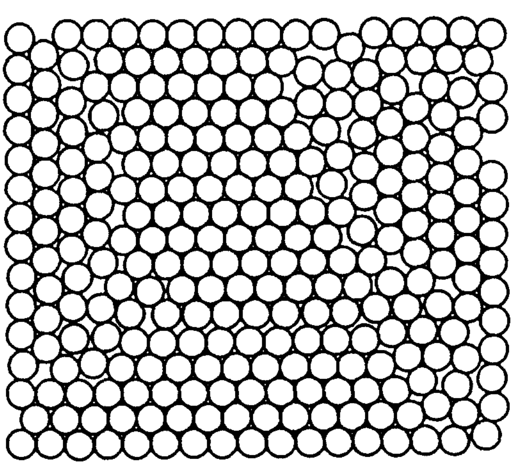


Fig. 2. Structure of packing used in the example.

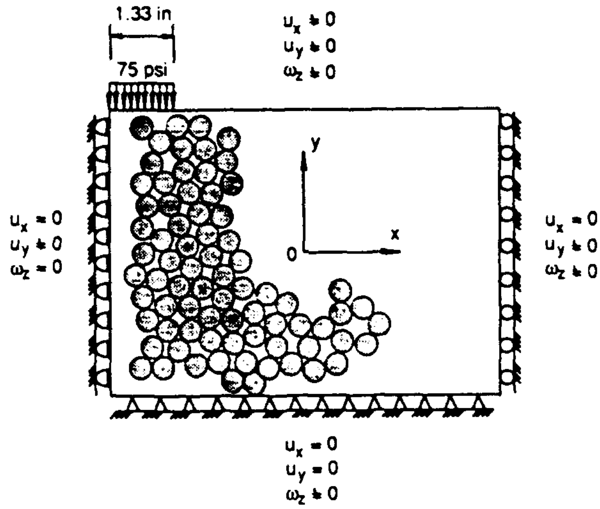


Fig. 3. Boundary condition of a granular medium subjected to surface loading.

The example shown in Fig. 3 represents a pressure load on a boundary surface. Because a symmetrical condition is assumed, Fig. 3 reveals only half of the picture on the right of the center line. The pressure is 75 lb in.$^{-1}$ (0.66 kN m^{-1}) loaded on a length of 1.33 in. (3.38 cm) on the surface. At the base, the displacements are specified to be zero in both directions. The displacements on the center line and on the side line are specified to be zero in the horizontal direction. All boundary points are free to rotate except for the points on the center line. The boundary conditions are shown schematically in Fig. 3.

Figure 4 is a finite element mesh to model the boundary value problem described in Fig. 3. The total area is divided into four elements. Each element is made of 16 nodal points. Since the geometric arrangement of particles that comprises each element is known and the properties of the grain contact are given, the element stiffness matrix can be computed based on eqn (58), as described in the previous section on the microstructural finite element method.

The element stiffness matrix for each element is then assembled into a global stiffness matrix. With the specified boundary condition, a set of simultaneous equations is formulated and solved for the displacements and rotations for all the nodal points. The solved nodal displacements and nodal rotations are used to obtain the movement of each particle using the interpolation function defined in eqn (53). Based on the movements of all particles, contact forces and contact couples are determined using eqns (1), (2) and (7). From the contact forces and contact couples, the stress for each particle point can be calculated using the definition of stress in eqn (31).

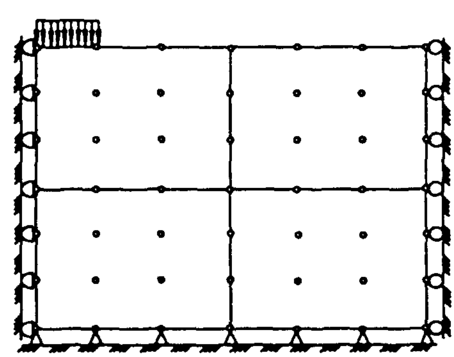
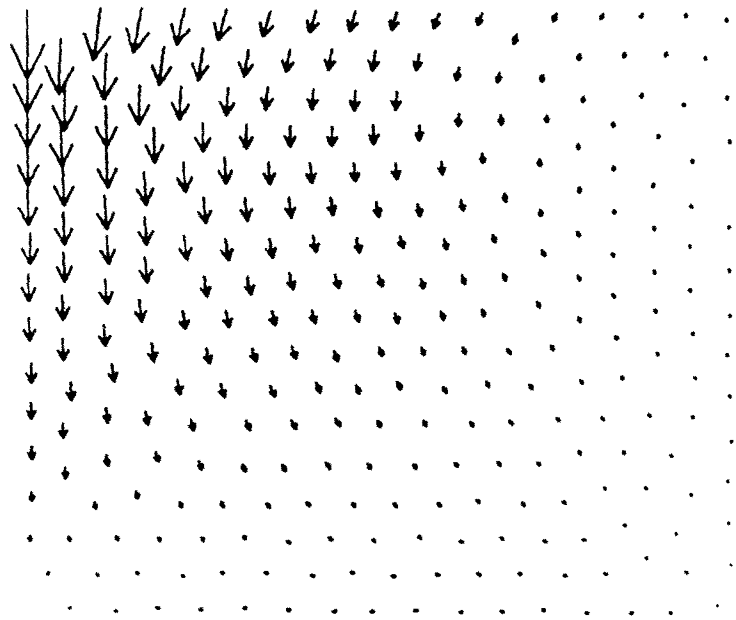
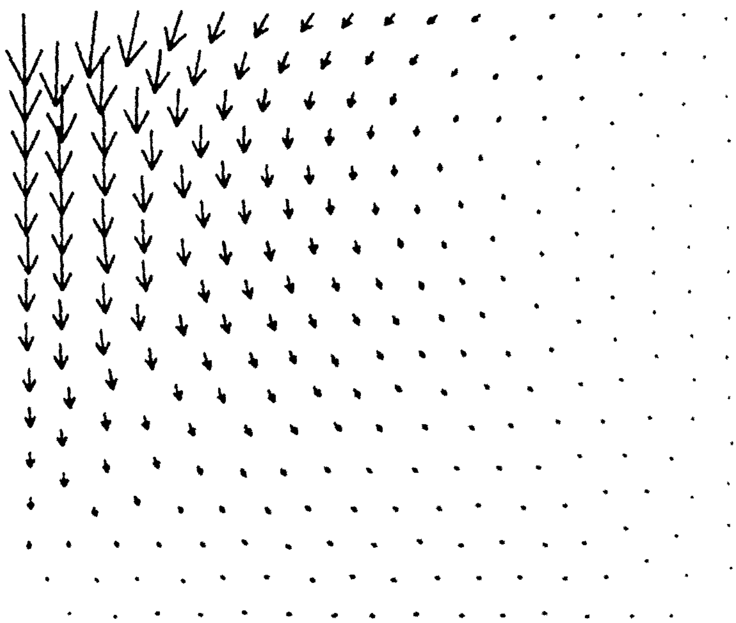


Fig. 4. Finite element mesh for the example.



(a) MFEM



(b) DEM

Scale : - 0.25 in

Fig. 5. Comparison of displacement fields obtained from Microstructure Finite Element Method (MFEM) and Discrete Element Method (DEM).

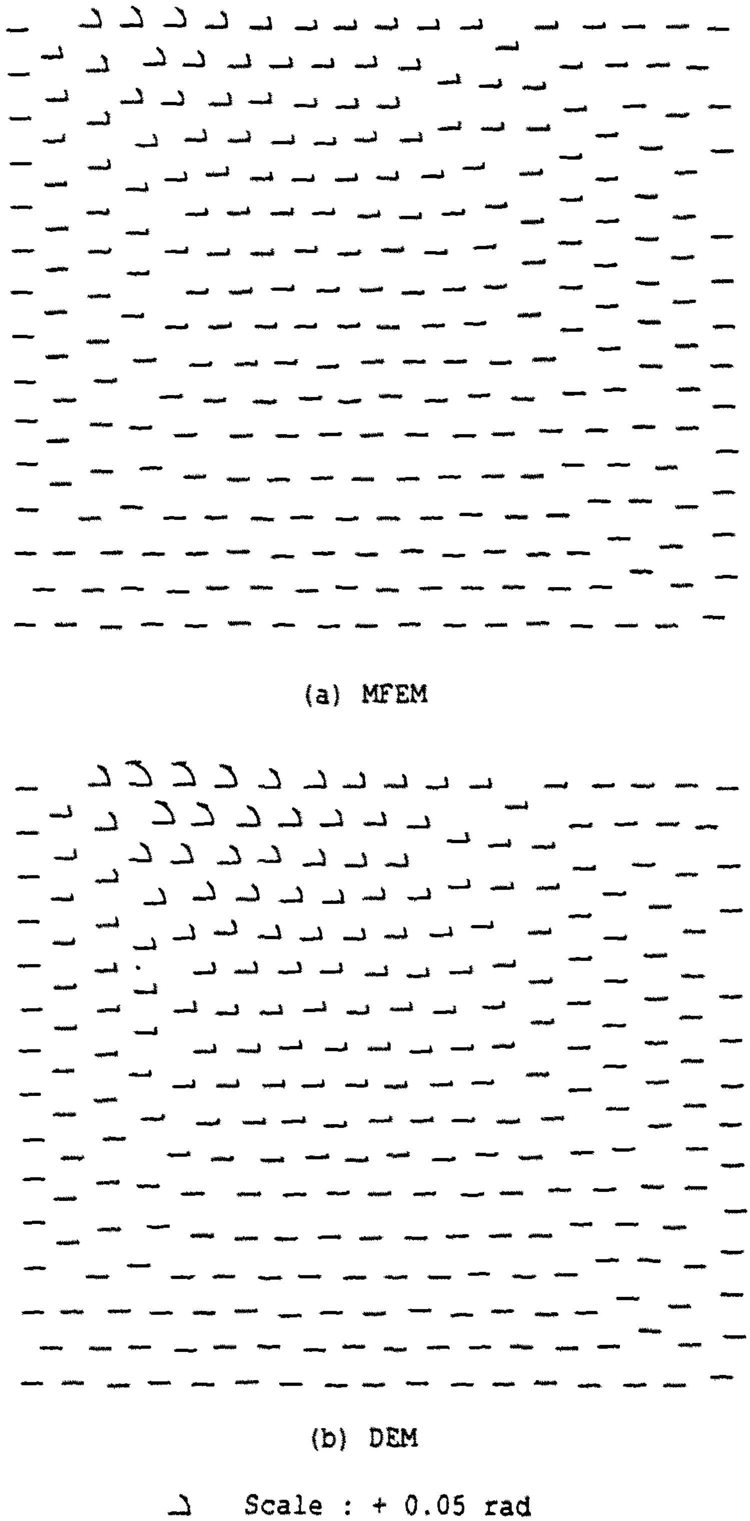


Fig. 6. Comparison of rotation fields obtained from Microstructure Finite Element Method (MFEM) and Discrete Element Method (DEM).

The above-mentioned computation procedure is carried out for the example problem. The computed displacement and rotation for each particle are plotted in Figs 5 and 6. Since the MFEM method is a continuum model, it is interesting to compare the results of MFEM with that obtained from computer simulation methods using entirely discrete models. Thus, the same example problem is also analyzed using the discrete element method (Chang and Misra, 1989), based on an earlier work by Serrano and Rodriguez-Ortiz (1973). The discrete element method (DEM) gives an exact solution because equilibrium is satisfied for each particle.

From the comparison shown in Figs 5 and 6, close agreement is found in the movement patterns for both particle displacements and particle rotations. However, a slightly larger magnitude is depicted for the results of DEM. For example, the settlement computed from DEM is 0.278 in. compared with 0.263 in. from MFEM.

The discrepancy is expected because DEM allows more degree of freedom since it solves a large set of simultaneous equations of the size of $3N \times 3N$, N being the total number of particles. On the other hand, the MFEM imposes constraints due to the discretization and it solves a smaller system of equations ($3M \times 3M$, M being the number of nodal points). This trade-off permits the MFEM to be a more efficient method for solving problems with a large number of particles.

In order to evaluate the effects of particle rotation and couple stress on the deformation behavior, we solve the example problem using MFEM method under two different conditions. The nodal rotations and the nodal couples are included in the first condition, whilst they are neglected in the second condition.

Differences in the computed displacement fields for the two conditions are reflected in Fig. 7. For the first case, where particle rotation is considered, particles near the surface move towards the center line and particles at depth move away from the center line. For the second case, where particle rotation is not considered, the results show that all particles have negligible horizontal movements. In general, the solution, allowing particle rotation, tends to give larger magnitudes of particle movements. For example, the settlements at the center line under the pressure load are 0.263 and 0.227 in., respectively, for the first and second cases.

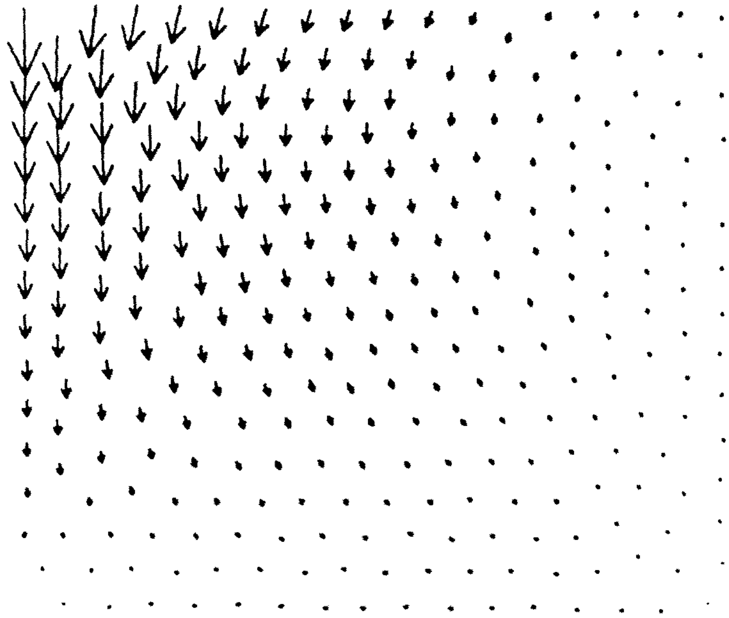
Although the vertical stress for particles, compared in Fig. 8, shows very close agreement between the results for both cases, a large discrepancy can be noted from the comparison of horizontal stress for particles shown in Fig. 9. Considering particle rotation and couple stress, the pressure load transmitted into the granular region diagonally causes compressive horizontal stress. In the region at some distance under the pressure load, the material experiences tensile horizontal stress. These phenomena are not found in the results for the case where particle rotation is not considered.

A large discrepancy is also found in the shear stress for particles from the comparison shown in Fig. 10. The pressure load caused more shear stress in the granular material due to the effect of particle rotation and the presence of couple stress. The distribution of couple stress computed is plotted in Fig. 11. Note that the large couple stresses are observed in the region adjacent to the pressure loading. This is inconsistent with the presence of a rotation gradient in this region, as previously shown in Fig. 6.

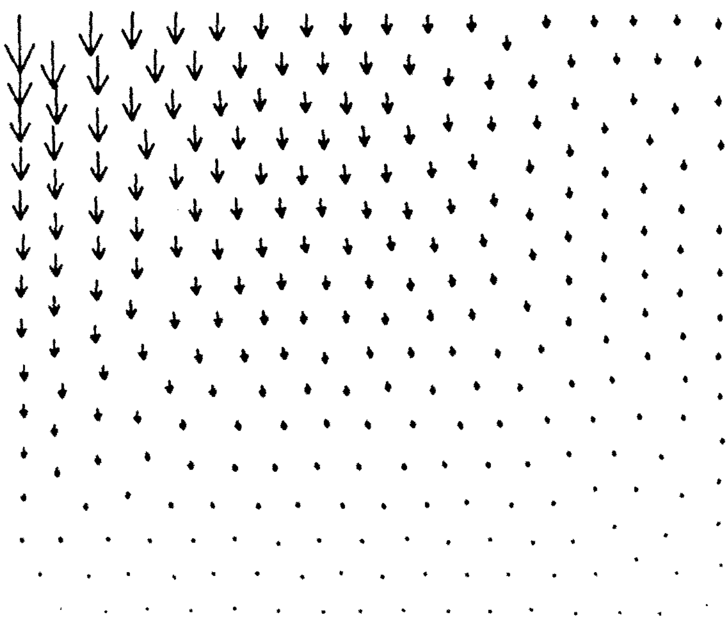
Based on the comparison of this example, particle rotation and couple stress have considerable effects on the overall deformation behavior of the assembly. It can be expected that such an effect should depend on the boundary value problem and the loading condition. In general, the couple stress plays a more important role when larger rotation gradients exist in the material.

5. SUMMARY AND CONCLUSIONS

Granular material, perceived as a collection of particles, is modelled as a macro-continuum taking account of the structural microdiscreteness of the material. In the model proposed, we treat particle translation and particle rotation as two independent continuum fields which lead accordingly to the definitions of stretch strain and polar strain of the material. Corresponding to these strains, we derive the relationships among contact forces,



(a) With effect of particle rotation



(b) Without effect of particle rotation

Scale : - 0.25 in

Fig. 7. Comparison of displacement fields with and without consideration of particle rotation using Microstructure Finite Element Method (MFEM).

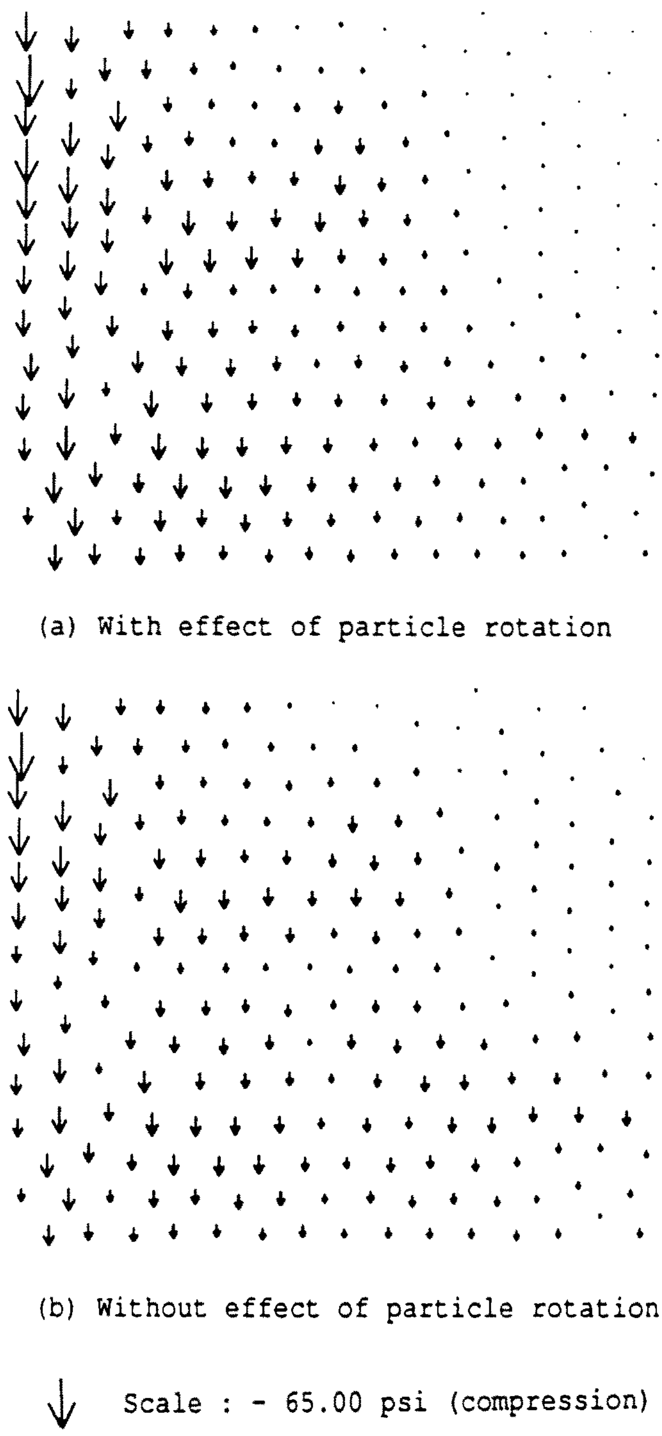


Fig. 8. Comparison of vertical stress fields with and without consideration of particle rotation using Microstructure Finite Element Method (MFEM).

contact moments, Cauchy stress and couple stress. Based on the particle interaction and grain contact properties, the stress–strain relationship for granular packing is established. A brief summary of the discrete variables and their analogue is given in Table 1.

The new feature of the present model, compared to the previous micropolar theories, is that the constitutive coefficients are expressed explicitly in terms of properties of grain contacts. This feature is particularly useful in the understanding of the constitutive coefficients associated with the couple stress, since these coefficients are difficult to measure experimentally in the laboratory tests from the gross behavior of a material sample.

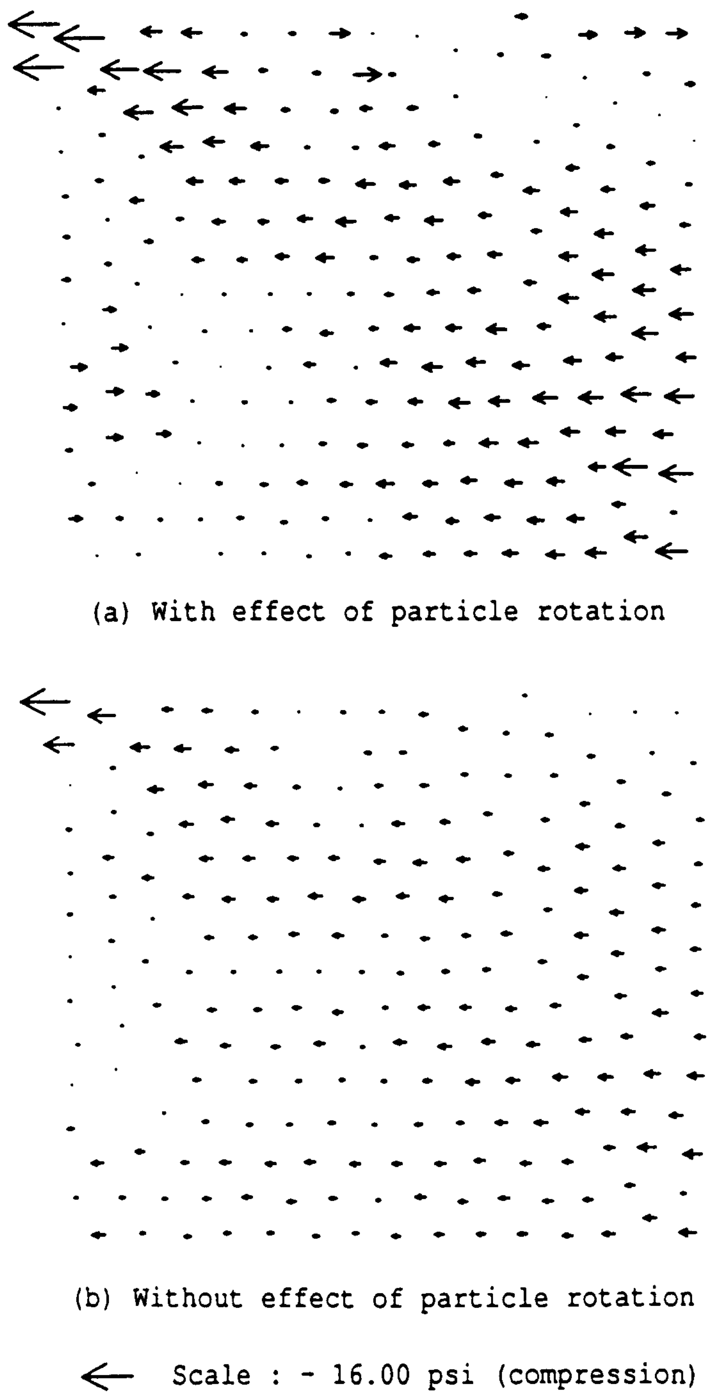


Fig. 9. Comparison of horizontal stress fields with and without consideration of particle rotation using Microstructure Finite Element Method (MFEM).

We have also presented a microstructural finite element procedure for the solution of boundary value problems. The results obtained from the example of a granular assembly under boundary pressure load have demonstrated the applicability of the proposed model.

Based on eqn (40), the ability of the couple stress to transmit through contact forces increases with particle size. Therefore, the couple stress is mostly transmitted through contact forces for materials made of larger size particles such as coarse sand and gravel. For materials made of small size particles, the couple stress is primarily transmitted through contact couples due to the rotational resistance of the contacts of the material such as cemented sand or silt.

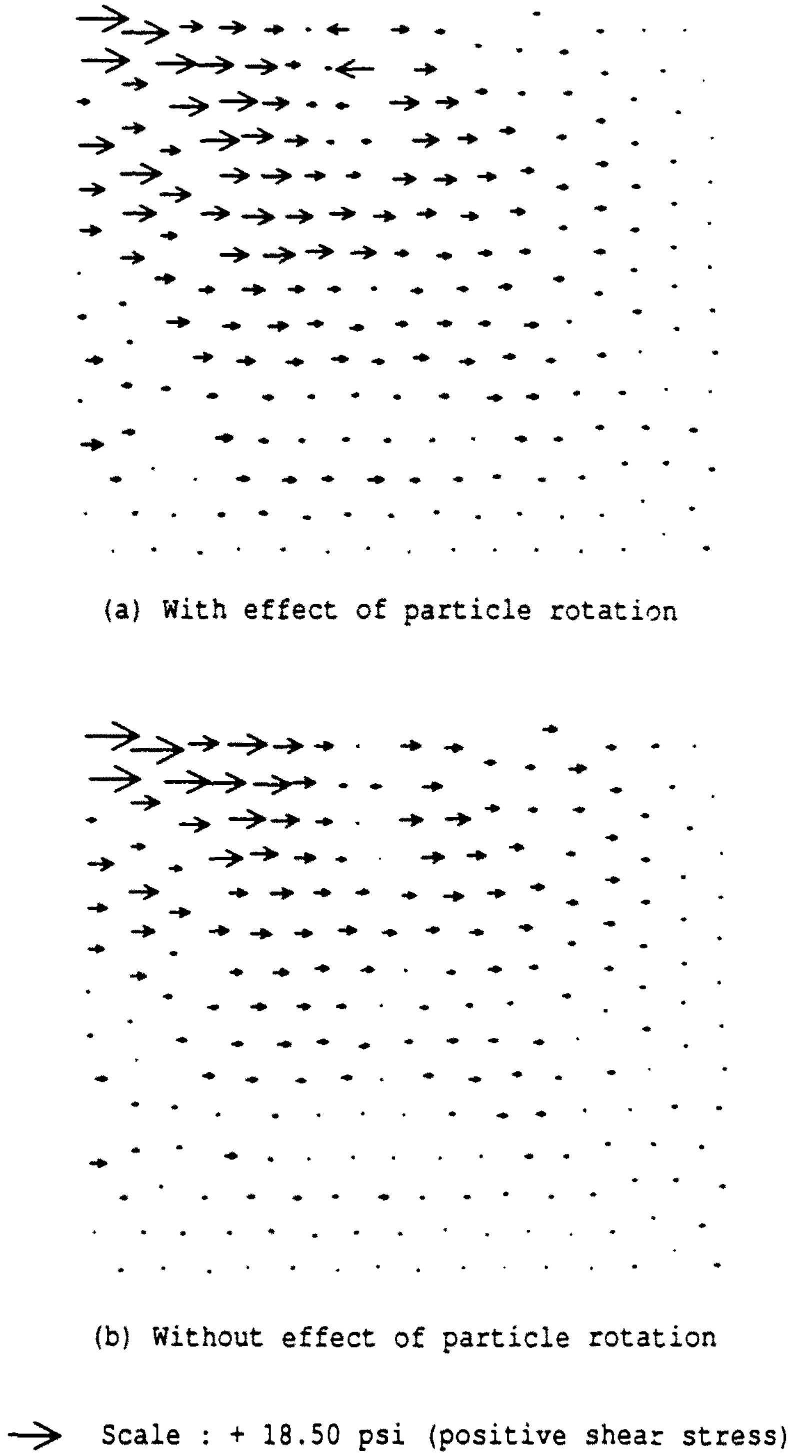


Fig. 10. Comparison of shear stress fields with and without consideration of particle rotation using Microstructure Finite Element Method (MFEM).

The effect of the couple stress on the deformation behavior of an assembly depends largely on the boundary and loading conditions. The couple stress has a significant effect on materials with large rotation gradients. On the other hand, the couple stress is negligible for conditions with small rotation gradients such as in the case of laboratory compression loading tests. For the example presented in this study, the effect of particle rotation and couple stress is significant on the overall deformation behavior of granular materials.

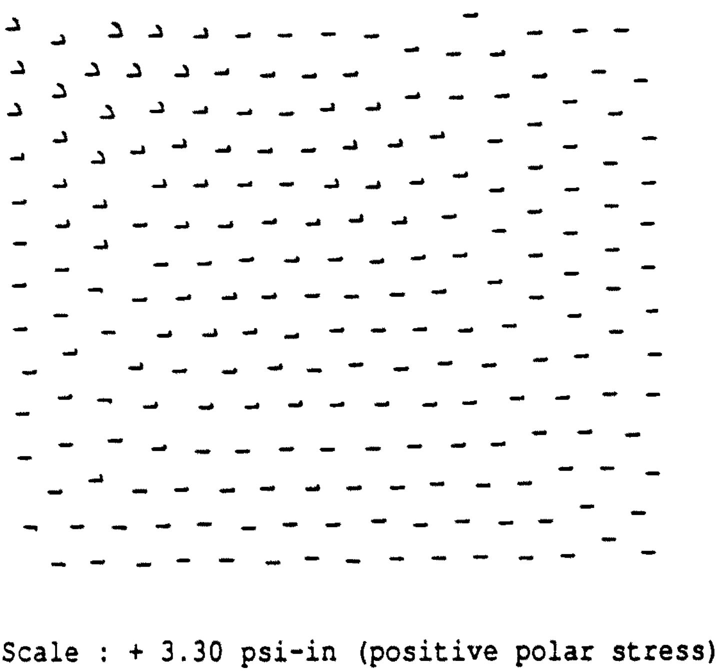


Fig. 11. Computed polar stress fields using Microstructure Finite Element Method (MFEM).

Table 1. Summary of discrete variables, the continuum analogues, and their relationships in granular material

Discrete variables		Equivalent continuum variables		Relationship
Force $\{F_i^c\}$	Contact force $\{f_i^c\}$ Contact moment $\{m_i\}$	Stress $\{S_{ij}\}$	Cauchy stress $\{\sigma_{ij}\}$ Polar stress $\{\mu_{ij}\}$	$\{S_{ij}\} = \frac{1}{2V}\sum [L_{ijk}^c]^T \{F_k^c\}$
Movement $\{U_i^a\}$	Translation $\{u_i^a\}$ Rotation $\{\omega_i^a\}$	Strain $\{E_{ij}\}$	Stretch strain $\{\varepsilon_{ij}\}$	$\{E_{ij}\} = [\nabla_{ijk}]\{U_k^a\}$
Rel. movement $\{D_i^c\}$	Rel. displ. $\{\delta_i^c\}$ Rel. rotation $\{\theta_i^c\}$		Polar strain $\{\gamma_{ij}\}$	$\{D_i^c\} = [L_{ijk}^c]\{E_{jk}\}$
Constitutive eqn $\{F_i^c\} = [K_{ij}^c]\{D_j^c\}$		Constitutive eqn $\{S_{ij}\} = [A_{ijkl}]\{E_{kl}\}$		$[A_{ijkl}] = \frac{1}{2V}\sum [L_{mij}^c][K_{mn}^c][L_{nkl}^c]$
Equilibrium eqn $\Sigma_c \{F_i^c\} = \{0\}$		Equilibrium eqn $[\nabla_{ijk}]^T\{S_{ik}\} = \{0\}$		
		Compatibility eqn $[\nabla_{jik}]\{E_{ik}\} - [\nabla_{lik}]\{E_{jk}\} = \{0\}$		

where gradient operator: $[\nabla_{jik}^c] = \begin{bmatrix} \partial_j \delta_{ik} & -e_{jik} \\ 0 & \partial_j \delta_{ik} \end{bmatrix}$

and fabric operator: $[L_{ijk}^c] = \begin{bmatrix} \delta_{ik} l_j^c & e_{ikl} l_j^c r_l^c \\ 0 & \delta_{ik} l_j^c \end{bmatrix}$

superscripts c : contact c
a : particle a.

REFERENCES

Ariman, T. (1972). Wave propagation in a micro-polar elastic half-space. *Acta mechanica* **13**(1-2), 11.

Besdo, D. (1985). Inelastic behaviour of plane frictionless block systems described as Cosserat media. *Arch. Mech.* **37**(6), 603-619.

Chang, C. S. (1987). Micromechanical modelling of constitutive relations for granular material. In *Micromechanics of Granular Materials* (Edited by M. Satake and J. T. Jenkins), pp. 271-279. Elsevier, Amsterdam.

Chang, C. S. and Liao, C. L. (1990). Constitutive relation for a particulate medium with the effect of particle rotation. *Int. J. Solids Structures* **26**(4), 437-453.

Chang, C. S. and Misra, A. (1989). Computer simulation and modelling of mechanical properties of particulates. *Comput. Geotechnics* **7**(4), 269–287.

Christoffersen, J., Mehrabadi, M. M. and Nemat-Nasser, S. (1981). A micromechanical description of granular material behavior. *ASME J. Appl. Mech.* **48**(2), 339–344.

Cosserat, E. and Cosserat, F. (1896). Sur la théorie de l'élasticité. *Ann. de L'Ecole Normale de Toulouse* **10**(1), 1.

Eringen, A. C. (1968). Theory of micropolar elasticity. In *Fracture—An Advanced Treatise* (Edited by H. Liebowitz), Vol. II, Chapter 7, pp. 621–693. Academic Press, New York.

Green, A. E. (1965). Micro-materials and multipolar continuum mechanics. *Int. J. Engng Sci.* **3**(5), 533.

Herrmann, G. and Achenbach, J. D. (1967). Application of theories of generalized Cosserat continua to the dynamics of composite materials. *IUTAM Symp.*, Freudenstadt-Stuttgart, *Mechanics of Generalized Continua* (Edited by E. Kroner). Springer, Berlin.

Kanatani, K. (1979). A micro-polar continuum theory for the flow of granular materials. *Int. J. Engng Sci.* **17**(4), 419–432.

Mindlin, R. D. (1962). Influence of couple stresses on stress concentration. *Exptl Mech.* **3**(1), 1–9.

Mindlin, R. D. (1965). Stress functions for a Cosserat continuum. *Int. J. Solids Structures* **1**(3), 265–271.

Muhlhaus, H. B. (1989). Application of Cosserat theory in numerical solutions of limit load problems. *Ing.-Arch.* **59**(2), 124–137.

Muhlhaus, H. B. and Vardoulakis, I. (1987). The thickness of shear bands in granular materials. *Geotechnique* **37**(3), 271–283.

Nowacki, W. (1986). *Theory of Asymmetric Elasticity*. PWN—Polish Scientific Publishers, Warsaw.

Serrano, A. A. and Rodriguez-Ortiz, J. M. (1973). A contribution to the mechanics of heterogeneous granular media. *Proc. Symp. on Plasticity and Soil Mechanics*, Cambridge, pp. 215–227.

Sternberg, E. (1968). Couple stresses and singular stress concentrations in elastic solid. Mechanics of generalized continua. *Proc. IUTAM Symp. on the Generalized Cosserat Continuum and the Continuum Theory of Dislocation with Applications* (Edited by E. Kroner). Springer, New York.

Toupin, R. A. (1964). Theory of elasticity with couple-stress. *Arch. Rational Mech. Anal.* **17**(2), 85–112.

Voigt, W. (1887). *Theoretische Studien über die Elastizitätsverhältnisse der Krystalle*, I, II. Abh. der Konigl. Ges. der Wiss, Göttingen, Vol. 34.